One Polynomial Mean Inequality.

Problem with a solution proposed by Arkady Alt , San Jose , California, USA.

Prove that for any real positive x and any natural $n \geq 2$ holds inequality

$$\sqrt[n]{\frac{1+x+\ldots+x^n}{n+1}} \geq \sqrt[n-1]{\frac{1+x+\ldots+x^{n-1}}{n}}, n \in \mathbb{N}$$

Solution.

First note that since $\sqrt[n]{\frac{1+x+\ldots+x^n}{n+1}} = x\sqrt[n]{\frac{1+1/x+\ldots+(1/x)^n}{n+1}}$ then

$$\sqrt[n]{\frac{1+x+\ldots+x^n}{n+1}} \geq \sqrt[n-1]{\frac{1+x+\ldots+x^{n-1}}{n}} \Leftrightarrow \sqrt[n]{\frac{1+1/x+\ldots+(1/x)^n}{n+1}} \geq \sqrt[n-1]{\frac{1+1/x+\ldots+(1/x)^{n-1}}{n}}$$

and we can assume $0 < x \leq 1$.

Let $S_n := 1+x+\ldots+x^n$. Then $\sqrt[n]{\frac{S_n}{n+1}} \geq \sqrt[n-1]{\frac{S_{n-1}}{n}} \Leftrightarrow \left(\frac{S_n}{n+1}\right)^{n-1} \geq \left(\frac{S_{n-1}}{n}\right)^n \Leftrightarrow$

(**1**) $\quad S_n^{n-1} n^n \geq S_{n-1}^n (n+1)^{n-1},\ n \geq 2.$

For $n = 2$ inequality (**1**) becomes $2^2 S_2 \geq 3S_1^2 \Leftrightarrow 4(1+x+x^2) \geq 3(1+x)^2 \Leftrightarrow (x-1)^2 \geq 0.$

Denoting $a_n := S_n^{n-1} n^n, b_n := S_{n-1}^n (n+1)^{n-1},\ n \geq 2$ we will prove auxiliary inequality $\frac{a_{n+1}}{a_n} \geq \frac{b_{n+1}}{b_n}$

We have $\frac{a_{n+1}}{a_n} \geq \frac{b_{n+1}}{b_n} \Leftrightarrow \frac{S_{n+1}^n (n+1)^{n+1}}{S_n^{n-1} n^n} \geq \frac{S_n^{n+1} (n+2)^n}{S_{n-1}^n (n+1)^{n-1}} \Leftrightarrow$

$S_{n+1}^n (n+1)^{n+1} \cdot S_{n-1}^n (n+1)^{n-1} \geq S_n^{n-1} n^n \cdot S_n^{n+1} (n+2)^n \Leftrightarrow$

$S_{n+1}^n S_{n-1}^n (n+1)^{2n} \geq S_n^{2n} (n^2+2n)^n \Leftrightarrow S_{n+1} S_{n-1} (n+1)^2 \geq S_n^2 (n^2+2n) \Leftrightarrow$

$(S_n + x^n)(S_n - x^n)(n+1)^2 \geq S_n^2 (n^2+2n) \Leftrightarrow S_n^2 (n+1)^2 - x^{2n}(n+1)^2 \geq S_n^2 (n^2+2n) \Leftrightarrow$

$S_n^2 \geq x^{2n}(n+1)^2 \Leftrightarrow S_n \geq x^n(n+1) \Leftrightarrow 1+x+\ldots+x^n \geq x^n(n+1) \Leftrightarrow 1+x+\ldots+x^{n-1} \geq nx^n$

where latter inequality obviously holds because $0 < x \leq 1 \Rightarrow x^k \geq x^n, k = 1,2,\ldots,n-1.$

For any natural $n \geq 2$ since $\frac{a_{n+1}}{a_n} \geq \frac{b_{n+1}}{b_n}, n \in \mathbb{N}$ then assuming $a_n \geq b_n$ we obtain

$a_{n+1} = a_n \cdot \frac{a_{n+1}}{a_n} \geq b_n \cdot \frac{b_{n+1}}{b_n} = b_{n+1}.$

Thus, by Math Induction for any $n \geq 2$ holds inequality $a_n \geq b_n \Leftrightarrow$(**1**).